Dynamic Entropy Pooling: Portfolio Management with Views at Multiple Horizons

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Background

The profit-and-loss (P&L)

The market model

Portfolio construction

Case studies
Background

The standard approach to discretionary portfolio management (Black-Litterman, Entropy Pooling) processes subjective views that refer to the distribution of the market at a specific single investment horizon.

The standard approach to multi-period portfolio management with market impact (Garleanu-Pedersen) processes non-discretionary (systematic) signals.

Dynamic Entropy Pooling is a quantitative approach to perform dynamic portfolio management with discretionary, multi-horizon views.

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- The standard approach to discretionary portfolio management (Black-Litterman, Entropy Pooling) processes subjective views that refer to the distribution of the market at a specific single investment horizon.
- The standard approach to multi-period portfolio management with market impact (Garleanu-Pedersen) processes non-discretionary (systematic) signals.
- Dynamic Entropy Pooling is a quantitative approach to perform dynamic portfolio management with discretionary, multi-horizon views.
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The profit-and-loss (P&L)

• We assume the single-period P&L is a set of exposures multiplied by the increments of the risk drivers over the rebalancing period:

\[ \Pi_{t+1} = b'_t \Delta X_{t+1} \]

• The set of risk drivers can be extended to include also external factors that do not affect directly the P&L of the instruments. On such additional factors we can express views that influence the P&L through correlation. The corresponding entries in the exposures vector will be set to zero.
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$$\Pi_{t+1} = b_t' \Delta X_{t+1}$$

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Consider an equity share or an index. Then the risk driver is its log-value:

$$X_t = \ln V_t$$

The P&L of a portfolio with $h_{n,t}$ shares in the $n$-th asset is:

$$\Pi_{t+1} = \sum_n h_{n,t} V_{n,t} \times \left( \frac{V_{n,t+1}}{V_{n,t}} - 1 \right) \approx \sum_n b_{n,t} \Delta X_{n,t+1}$$

More in general, in terms of a style/risk linear factor model:

$$\Pi_{t+1} = \sum_k b_{k,t}^{style} \Delta X_{k,t+1}^{style}$$
- We assume the single-period P&L is a set of exposures multiplied by the increments of the risk drivers over the rebalancing period:
  \[ \Pi_{t+1} = b'_t \Delta X_{t+1} \]

- The set of risk drivers can be extended to include also external factors that do not affect directly the P&L of the instruments. On such additional factors we can express views that influence the P&L through correlation. The corresponding entries in the exposures vector will be set to zero.

- Suppose that the \( n \)-th asset is a fixed income instrument. Its value at the first order satisfies
  \[ \Pi_{n,t+1} \approx -\sum_k dv01_{n,k,t} \Delta Y_{k,t+1} \]
  where \( Y_{k,t} \) is the \( k \)-th key-rate on the yield curve; \( dv01_{n,k,t} \) is the dollar-sensitivity of the \( n \)-th instrument to \( Y_{k,t} \).

- Then the P&L due to a set of fixed income instruments is:
  \[ \Pi_{t+1} \approx \sum_k \left( -\sum_n h_{n,t} dv01_{n,k,t} \right) b_{k,t} \Delta X_{k,t+1} \]
The profit-and-loss (P&L)

- We assume the single-period P&L is a set of exposures multiplied by the increments of the risk drivers over the rebalancing period:

\[ \Pi_{t+1} = b'_t \Delta X_{t+1} \]

- The set of risk drivers can be extended to include also external factors that do not affect directly the P&L of the instruments. On such additional factors we can express views that influence the P&L through correlation. The corresponding entries in the exposures vector will be set to zero.

For a stock option, the risk drivers are the log-value of the underlying and the implied volatility \( X_t = \ln V_t \) and \( \sum_t^{\text{impl}} \)

- Then for a portfolio of stock options, the P&L is:

\[ \Pi_{t+1} \approx \sum_n \left( h_{n,t} \delta_{n,t} V_{n,t} \Delta X_{n,t+1} + h_{n,t} \nu_{n,t} \Delta \sum_t^{\text{impl}} \right) b_{\delta_{n,t}} + b_{\nu_{n,t}} \]

where \( \delta_{n,t} \) and \( \nu_{n,t} \) are the delta and vega of the \( n \)-th option.
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• Consider a book of assets driven by a set of \( \bar{n} \) risk drivers \( \mathbf{X}_t \) (interest rates, implied volatility surfaces, log-prices, etc.)

• We assume that the drivers follow a MVOU process:

\[
dX_t = (-\theta \mathbf{X}_t + \mu) \, dt + \sigma d\mathbf{W}_t
\]

• Choose a set of discrete monitoring dates \( t, t + 1, \ldots, \bar{t} \)

• Stack the process at the monitoring times as follows:

\[
\mathbf{X}_{t \rightarrow \bar{t}} \equiv \begin{pmatrix} \mathbf{X}_t \\ \mathbf{X}_{t+1} \\ \vdots \\ \mathbf{X}_{\bar{t}} \end{pmatrix}
\]

• Then the process is jointly multivariate normal at all times

\[
\mathbf{X}_{t \rightarrow \bar{t}} | \mathbf{i}_t \sim N(\mu_{t \rightarrow \bar{t}}, \sigma^2_{t \rightarrow \bar{t}})
\]
Market model

• The expectation vector of the is

\[
\mu_{t \rightarrow \tilde{t}} \equiv \left( e^{-0\theta} x_t + (\mathbb{I}_n - e^{-0\theta}) \theta^{-1} \mu, e^{-1\theta} x_t + (\mathbb{I}_n - e^{-1\theta}) \theta^{-1} \mu, e^{-\bar{t} - t}\theta x_t + (\mathbb{I}_n - e^{-\bar{t} - t}\theta) \theta^{-1} \mu \right)
\]

• The covariance matrix is

\[
\sigma_{t \rightarrow \tilde{t}}^2 \equiv \left( \begin{array}{cccc}
\sigma_0^2 & \sigma_0^2 e^{-\theta'} & \sigma_0^2 e^{-2\theta'} & \sigma_0^2 e^{-\bar{t} - t}\theta' \\
\sigma_0 e^{-\theta} \sigma_0^2 & \sigma_1^2 & \sigma_1^2 e^{-\theta'} & \sigma_1^2 e^{-\bar{t} - t - 1}\theta' \\
e^{-2\theta} \sigma_0^2 & \sigma_1 e^{-\theta} \sigma_1^2 & \sigma_2^2 & \cdot \\
e^{-\bar{t} - t}\theta \sigma_0^2 & e^{-\theta} \sigma_1^2 & e^{-\bar{t} - t}\theta \sigma_2^2 & \sigma_{\bar{t} - t}^2
\end{array} \right)
\]

where

\[
vec(\sigma^2_{\tau}) \equiv (\theta \oplus \theta)^{-1} \left( \mathbb{I}_n^2 - e^{-(\theta \oplus \theta)\tau} \right) vec(\sigma^2)
\]

Dynamic Entropy Pooling: Portfolio Management with Views at Multiple Horizons
We extend the Entropy Pooling approach in Meucci (2010) to the case of multiple horizons

- **The prior**: assume a model for the joint distribution of the process at the monitoring times:

  \[ X_{t \rightarrow t'} | i_t \sim f \]

- **The views**: are statements (constraints) on the yet-to-be defined distribution of the process:

  \[ g \in \mathcal{V}_t \]

- **The posterior**: is the closest distribution to the prior that satisfies the views:

  \[ \overline{f} \equiv \arg\min_{g \in \mathcal{V}_t} \{ \mathcal{E}(g, f) \} \]

  where the “distance” is the relative entropy

  \[ \mathcal{E}(g, f) \equiv \int g(x_t, \ldots, x_{t'}) \ln \frac{g(x_t, \ldots, x_{t'})}{f(x_t, \ldots, x_{t'})} \, dx_t \cdots dx_{t'} \]
We extend the Entropy Pooling approach in Meucci (2010) to the case of multiple horizons

- **The prior**: assume a MVOU model for the joint distribution of the process at the monitoring times

\[ X_{t \sim \bar{t}} \mid i_t \sim N(\mu_{t \sim \bar{t}}, \sigma_{t \sim \bar{t}}^2) \]

- **The views**: are statements (constraints) on the yet-to-be-defined distribution of the process:

\[ \mathcal{V}_t : \left\{ \begin{array}{l} \mathbb{E}_t^g \{ \nu_{t \mu} X_{t \sim \bar{t}} \} \equiv \mu_{\text{view};t} \\
\mathbb{C}_t^g \{ \nu_{t \sigma} X_{t \sim \bar{t}} \} \equiv \sigma_{\text{view};t}^2. \end{array} \right. \]

where \( \nu_{t \mu} \) and \( \nu_{t \sigma} \) are matrices that defines arbitrary linear combinations of the process at the times for the views.

- **The posterior**: is the closest distribution to the prior that satisfies the views:

\[ \bar{f} \equiv \arg\min_{g \in \mathcal{V}_t} \{ \mathcal{E}(g, f) \} \Rightarrow X_{t \sim \bar{t}} \mid i_t \sim N(\bar{\mu}_{t \sim \bar{t}}, \bar{\sigma}_{t \sim \bar{t}}^2) \]
Market model

\[ X_{t\rightarrow t} | i_t \sim N(\mu_{t\rightarrow t}, \sigma_{t\rightarrow t}^2) \]

- For the expectation, we introduce the pseudo inverse matrix of \( \mathbf{v}_{\mu,t} \)

\[ \mathbf{v}_{\mu,t}^+ \equiv \sigma_{t\rightarrow t}^2 \mathbf{v}_{\mu,t}'(\mathbf{v}_{\mu,t} \sigma_{t\rightarrow t}^2 \mathbf{v}_{\mu,t}')^{-1} \]

we define the two complementary projectors:

\[ \mathbb{P}_{\mu,t} \equiv (\mathbb{I}_{\bar{n}(\bar{t}-t+1)} - \mathbf{v}_{\mu,t}^+ \mathbf{v}_{\mu,t}) \quad \mathbb{P}^\perp_{\mu,t} \equiv \mathbf{v}_{\mu,t}^+ \mathbf{v}_{\mu,t} \]

Then

\[ \mu_{\mu_{t\rightarrow \bar{t}}} \equiv \mathbb{P}_{\mu,t} \mu_{t\rightarrow \bar{t}} + \mathbb{P}^\perp_{\mu,t}(\mathbf{v}_{\mu,t}^+ \mathbf{v}_{\mu,t}) \]

- Similar, for the covariance we introduce the pseudo inverse of \( \mathbf{v}_{\sigma,t} \)

\[ \mathbf{v}_{\sigma,t}^+ \equiv \sigma_{t\rightarrow \bar{t}}^2 \mathbf{v}_{\sigma,t}'(\mathbf{v}_{\sigma,t} \sigma_{t\rightarrow \bar{t}}^2 \mathbf{v}_{\sigma,t}')^{-1} \]

and the two complementary projectors:

\[ \mathbb{P}_{\sigma,t} \equiv \mathbb{I}_{\bar{n}(\bar{t}-t+1)} - \mathbf{v}_{\sigma,t}^+ \mathbf{v}_{\sigma,t} \quad \mathbb{P}^\perp_{\sigma,t} \equiv \mathbf{v}_{\sigma,t}^+ \mathbf{v}_{\sigma,t} \]

Then

\[ \sigma_{t\rightarrow \bar{t}}^2 \equiv \mathbb{P}_{\sigma,t} \sigma_{t\rightarrow \bar{t}}^2 \mathbb{P}_{\sigma,t} + \mathbb{P}^\perp_{\sigma,t}(\mathbf{v}_{\sigma,t}^+ \mathbf{v}_{\sigma,t}^2 \mathbf{v}_{\sigma,t}^2)(\mathbb{P}^\perp_{\sigma,t})' \]
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As in Garleanu and Pedersen (2013), the satisfaction functional is an infinite sum of discounted trade-offs:

$$S_t^{(\gamma, \eta)} \equiv \sum_{s=t}^{\infty} e^{-\lambda(s-t)} \left[ \mathbb{E}\left\{ \prod_{(s,s+1]} I_t \right\} - \frac{\gamma}{2} \mathbb{V}\left\{ \prod_{(s,s+1]} I_t \right\} - \frac{\eta}{2} \mathbb{E}\left\{ MI_s | I_t \right\} \right]$$

where the market impact is a quadratic function of the exposure rebalancing

$$MI_t = a^2 + \Delta b_t' c^2 \Delta b_t$$

with $c^2$ a suitable positive definite matrix. Note the term $a^2$, which represents the average cost of maintaining constant exposures
Given that the P&L is linear in the exposures $\Pi_{t+1} = b'_t \Delta X_{t+1}$, we need to solve for the optimal policy of exposures as functions of information:

$$\{b^*_s = p^*_s(i_s)\}_{s \geq t}$$

where

$$\{p^*_s\}_{s \geq t} = \operatorname{argmax}_{\{p_s\}_{s \geq t} \in \mathcal{C}} \mathbb{E}_t \left\{ \sum_{s=t}^{\infty} e^{-\lambda(s-t)} [p_s(I_s)' \omega \mathbb{E}_s \{\Delta X_{s+1}\} - \frac{\gamma}{2} p_s(I_s)' \omega \mathbb{C} v_s \{\Delta X_{s+1}\} \omega' p_s(I_s) - \frac{\eta}{2} \Delta p_s(I_s)' c^2 \Delta p_s(I_s)] \right\}$$

As in Garleanu and Pedersen (2013), the satisfaction functional is an infinite sum of discounted trade-offs:

$$\overline{S}_t^{(\gamma, \eta)} \equiv \sum_{s=t}^{\infty} e^{-\lambda(s-t)} [\mathbb{E}\{\Pi_{(s,s+1)}|i_t\} - \frac{\gamma}{2} \mathbb{V}\{\Pi_{(s,s+1)}|i_t\} - \frac{\eta}{2} \mathbb{E}\{MI_s|i_t\}]$$

where the market impact is a quadratic function of the exposure rebalancing

$$MI_t = a^2 + \Delta b'_t c^2 \Delta b_t$$

with $c^2$ a suitable positive definite matrix. Note the term $a^2$, which represents the average cost of maintaining constant exposures.
• Given that the P&L is linear in the exposures $\Pi_{t+1} = b'_t \Delta X_{t+1}$, we need to solve for the optimal policy of exposures as functions of information

$$\{b^*_s = p^*_s(i_s)\}_{s \geq t}$$

where

$$\{p^*_s\}_{s \geq t} = \arg\max \{p_s\}_{s \geq t} \in \mathbb{C} \mathbb{E}_t \left\{ \sum_{s=t}^{\infty} e^{-\lambda(s-t)} [p_s(I_s)'\omega \mathbb{E}_s\{\Delta X_{s+1}\} - \frac{\gamma}{2} p_s(I_s)'\omega \mathbb{E}_s\{\Delta X_{s+1}\}\omega' p_s(I_s) - \frac{\eta}{2} \Delta p_s(I_s)'c^2 \Delta p_s(I_s)] \right\}$$

• Dynamic programming with a quadratic value function yields a recursive problem with time-dependent coefficients

$$v_{s+1}(b_s, x_{s+1}) = -\frac{1}{2} b'_s \psi_{bb,s} b_s + b'_s \psi_{bx,s} x_{s+1} + \frac{1}{2} x'_{s+1} \psi_{xx,s} x_{s+1} + \psi'_b b_s + \psi'_x x_{s+1} + \psi_{0,s}$$

$$\iff \psi_{s-1} = g_s(\psi_s)$$

• The optimal policy of exposures then reads

$$b^*_s = \left( \frac{\gamma \omega \sigma^2_s \omega' + \eta c^2 + e^{-\lambda} \psi_{bb,s}}{\eta c^2 b_{s-1}} \right)^{-1} \left[ \frac{\eta c^2 b_{s-1}}{\text{legacy exposures}} + (\omega \beta_s + e^{-\lambda} \psi_{bx,s}(\beta_s + \Pi_n)) x_s + (\omega + e^{-\lambda} \psi_{bx,s}) \alpha_s + e^{-\lambda} \psi_{b,s} \right]$$

- current risk drivers
- (\#) future views
Dynamic Entropy Pooling: Portfolio Management with Views at Multiple Horizons

Portfolio construction

- Given that the P&L is linear in the exposures $\Pi_{t+1} = b'_t \Delta X_{t+1}$, we need to solve for the optimal policy of exposures as functions of information

$$\{b^*_s = p^*_s(i_s)\}_{s \geq t}$$

where

$$\{p^*_s\}_{s \geq t} = \text{argmax}_{\{p_s\}_{s \geq t}} \mathbb{E}_t \{ \sum_{s=t}^{\infty} e^{-\lambda(s-t)} [p_s(I_s)' \omega \mathbb{E}_s \{\Delta X_{s+1}\} - \frac{\gamma}{2} p_s(I_s)' \omega \mathbb{E}_s \{\Delta X_{s+1}\} \omega' p_s(I_s) - \frac{\eta}{2} \Delta p_s(I_s)' c^2 \Delta p_s(I_s)] \}$$

- With no market impact, we obtain a series of myopic one-period problems
- The optimal policy is a sequence of mean-variance optimizations based on the posterior distribution of the risk drivers process

$$b^*_s = \frac{1}{\gamma} (\omega \sigma_s^2 \omega')^{-1} \omega (\mathbb{P}_{\mu, s})_{s+1, \Delta \mu_{s \to t}} L o n g T e r m \quad \text{Long Term}$$

$$b^*_s = \frac{1}{\gamma} (\omega \sigma_s^2 \omega')^{-1} \omega (\mathbb{P}_{\mu, s})_{s+1, \Delta \mu_{s \to t}} View Mean$$

$$b^*_s = \frac{1}{\gamma} (\omega \sigma_s^2 \omega')^{-1} \omega (\mathbb{P}_{\mu, s})_{s+1, \Delta \mu_{s \to t}} View Mean$$

$$\mathbb{V}_{\mu, s} \mu_{\text{view};s} - x_s$$

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- The profit-and-loss (P&L)

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Case studies

One risk driver, one view
Case studies

Two risk drivers (one investable), two views

[Graphs showing exposure and loss rate for different risk drivers and views, with comparisons and simulations.]